Extensional Flow of a Two-Phase Fiber

A steady, isothermal flow of a two-phase fiber spinning has been studied. To investigate the spinline velocity profile affected by the interaction between two fluids with different rheological properties, a Newtonian and an upper-convected Maxwell fluid were considered to be the core and skin layers, respectively. Unlike the one-dimensional analysis, which has been the typical approach for spinning flows, radial dependency of the flow field was maintained in our analysis. Consequently, a set of partial differential equations were derived for the flow field in the draw-down region where lubrication scalings are applicable. Asymptotic solutions were then obtained for two limiting cases, where the Deborah number of the skin layer is small and the applied axial tension is large. When the Deborah number of the skin layer is small, a velocity correction to the Newtonian profile indicates a strong effect of the flow rate ratio as well as the shear viscosity ratio of the two fluids. For the case of a large applied tension, which is more relevant to industrial practice, very interesting flow behaviors are predicted. In this limit, the viscoelasticity effect of the skin layer is shown to be dominant and dictates the mechanics of the flow even when both its flow rate and shear viscosity may be much smaller than those of the Newtonian core layer. The result also predicts that hoop stress has a negligible effect on the axial velocity profile.

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Introduction

Extensional flow of polymer melts is an essential part of polymer processing as it commonly occurs in many fabrication processes such as fiber spinning, blown film extrusion, slot cast, extrusion coating, and blow molding. Besides the fact that stability of such extensional flows is of great importance, strain history of the polymer melts has strong influence on the physical properties of the final products (Dees and Spruiell, 1974; Bell and Edie, 1987). Thus, proper understanding of the mechanics of the flow is very important in industrial practice.

Among various extensional flows, fiber spinning may have been the subject of most extensive studies probably due to its simple geometry. Although the deformation of the polymer melt in a fiber spinning process is inhomogeneous, variation of the fiber radius in the spinning direction is generally very slow. Thus, it may be assumed that the flow is locally a uniform, uniaxial extension. Under this assumption, Matovich and Pearson (1969) studied isothermal flows of a Newtonian fluid, a power-law fluid, and a second-order fluid. In the case of a Newtonian fluid, if only viscous force is important, their result shows that the axial velocity profile is exponential. Pearson and Matovich (1969), in their investigation of the stability of the exponential profile to infinitesimal distrubances, predicted that the system becomes unstable if the draw ratio, which is the ratio of take-up velocity to initial velocity, is greater than about 20. The instability is often referred to as "draw resonance," in which the radius of the fiber oscillates periodically. Yeow (1974) applied the same approach to a flow of a sheet to model a slot cast process, which is a two-dimensional counterpart of the axisymmetric fiber spinning, and derived the same critical draw ratio of about 20.

Denn et al. (1975) studied spinning of a viscoelastic liquid using a generalized upper-convected Maxwell model. Their analysis predicted a linear velocity profile and high stress levels in spinning that match qualitatively with the features of polymeric fluids observed in many experiments. Fisher and Denn (1976) extended the study by including the deformation rate dependency of polymeric materials (White-Metzner model) and performed a linear stability analysis that predicted an upper stable region at a very high draw ratio. Their analyses served as a basis for many studies that followed using various viscoelastic constitutive equations such as corotational Maxwell model, Oldroyd-B fluid model, and Doi-Edwards model (Petrie, 1979;

Phan-Thien and Caswell, 1986; Larson, 1983). A very thorough survey of numerous studies on the subject of fiber spinning and its stability can be found in Petrie and Denn (1976), and more recent review articles have been provided by Denn (1980, 1983) and Mewis and Petrie (1987).

While there exist a significant amount of work on the "single-phase" spinning flow, the corresponding "multiphase" flow has not been treated. A multiphase extensional flow refers to a flow situation in a coextrusion process, and a schematic of a two-phase coextrusion fiber spinning is given in Figure 1. Although coextrusion is known for many years and quite prevalent in plastics industry, there exist relatively few fundamental studies about the subject (Han, 1975; Waters and Keeley, 1987; Nordberg and Winter, 1988). Furthermore, most of the existing studies focused on shear flow behaviors, and the extensional flow has not been considered at all despite its importance in industrial practice. Thus, we study the extensional flow of a two-phase fiber in this paper. Freeman et al. (1986) studied the drawing of hollow tubes from molten polymers and showed the effect of internal pressure on the ratio of inner to outer radii of the hollow tube. While hollow tubes may have resemblance to two-phase fibers, drawing of hollow tubes appears to be physically different from the spinning of two-phase fibers as will be discussed.

Based on many studies of single-phase fiber spinning, we may anticipate that the mechanics of a multiphase flow will be different from those of the single-phase flows of each component. The difference may be more pronounced if the rheological properties of the components are quite different from one another. For example, fundamental differences in the flow behaviors of

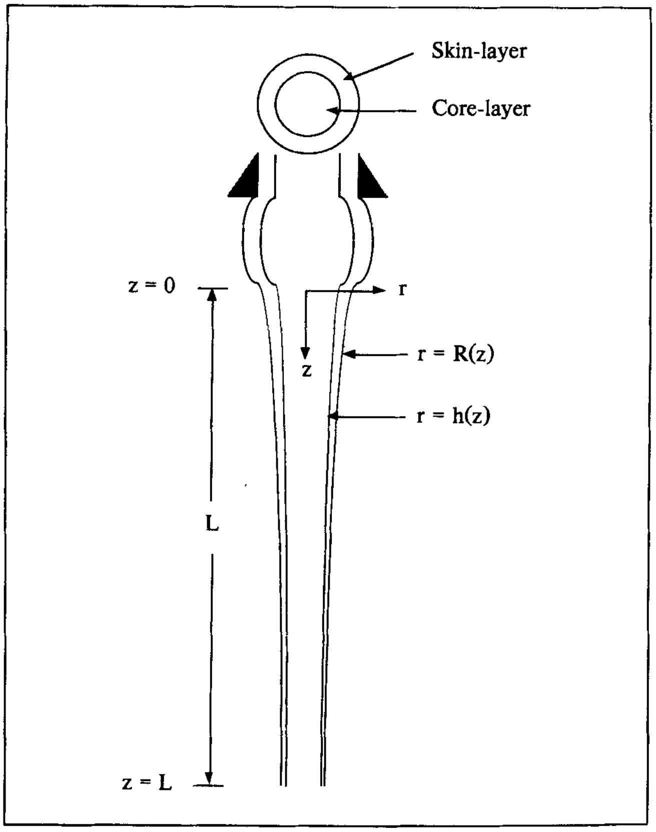

Figure 1. Coextrusion fiber spinning.

branched polyethylene and linear polyethylene are well known through many experimental studies (Cogswell, 1969; Chang and Lodge, 1972; Spearot and Metzner, 1972; Minoshima and White, 1986). Branched materials, unlike linear ones, often show extension hardening behavior in which extensional viscosity increases with deformation and deformation rate. They usually have a larger relaxation time than the linear materials and tend to have better spinning stability in terms of draw resonance probably due to the stabilizing effect of the viscoelasticity. But their spinnability is sometimes limited by breakage, as the stress in the melt may exceed the cohesive strength of the materials. If those two materials are coextruded, the mechanics of the two-phase flow may be affected strongly by the interaction between the rheological properties of the two fluids.

As the difference in relaxation times between the two materials may have the strongest effect on the mechanics of the flow in an extensional flow situation, we take a simple model for a two-phase flow, in which a Newtonian fluid and an upper-convected Maxwell fluid are the core layer and the skin layer, respectively. The major objective of our study is to investigate how the mechanics of the two-phase flow is affected by the rheological properties of each component. Since our analysis encompasses the single-phase flows of a Newtonian and an upper-convected Maxwell fluid as two special cases, the results can be compared with those of Matovich and Pearson (1969) and Denn et al. (1975).

Current study, like many others, focuses on the draw-down region where the fluids are stretched forming a thin fiber. We also assume that the variation of the fiber radius is very slow in the axial direction. Thus, an asymptotic expansion is applied in a small parameter ϵ, which is defined as the ratio of the radial characteristic length scale to the axial one. In the analyses of both Matovich and Pearson (1969) and Denn et al. (1975), *a priori* assumptions were made that the axial velocity, pressure, and stress fields are independent of the radial coordinate. Consequently, their analyses were one-dimensional at the outset. In our study, however, such assumptions are not made, and a standard formal expansion procedure is carried out while retaining the radial dependency of all variables.

The same expansion procedure was applied previously to the spinning of Newtonian fluids as well as Maxwell fluids by Schultz and Davis (1982) and Schultz (1987), respectively. Schultz and Davis recovered the results of Matovich and Pearson (1969) as a leading order solution and proceeded to obtain higher-order correction terms. In the work on slender viscoelastic fiber flow, Schultz (1987) studied a flow with a weak viscoelasticity effect and concluded that there is a serious limitation to the one-dimensional theory of Denn et al. (1975). Since the approach of Schultz is very similar to ours, his work can be closely compared with our development. As will be discussed, however, his contention regarding the limitation of the one-dimensional theory appears to be incorrect.

The results of our analysis show that the axial velocity and pressure field of the Newtonian core layer are independent of the radial coordinate at the leading order in ϵ. For the viscoelastic skin layer, however, there is no clear indication that pressure and stress fields are radially independent, although the axial velocity is. Consequently, unlike the one-dimensional analysis, a set of partial differential equations is derived for the stress field of the skin layer coupled with the radially uniform axial velocity. Since the equations that allow the radial dependency of the stress field have not been discussed previously, the derivation is briefly described here.

For the analysis of the partial differential equations, two limiting cases are considered, in which the viscoelasticity effect of the skin-layer is very small (the case of a small Deborah number) and very large (the case of a large applied axial tension), respectively. When the Deborah number of the skin layer is small, the lowest-order problem is identical to the flow of two Newtonian fluids, and the leading order velocity profile is exponential regardless of the shear viscosity ratio or the flow rate ratio of the two fluids. Higher-order corrections are, however, strongly affected by the shear viscosity and the flow rate ratios. The case of a large applied axial tension is more relevant to industrial practice and draws some results with important practical implications. In this limit, the viscoelasticity effect of the skin-layer becomes dominant and dictates the mechanics of the flow even when the flow rate and the shear viscosity of the skin layer are much smaller than those of the Newtonian core layer. It is also shown that hoop stress has no effect on the velocity profile in this limit, while it affects the radial dependency of the pressure field. It is interesting to note that the insensitivity of the velocity profile to the hoop stress was also predicted for drawing of hollow tubes by Freeman et al. (1986).

Problem Formulation

Figure 1 shows a schematic of the two-phase extensional flow. The core layer and skin layer are a Newtonian and an upper-convected Maxwell fluid, respectively. The flow is assumed to be isothermal and axisymmetric. To make an analytic progress, our analysis will be limited to the draw-down region where the variation of the fiber radius is assumed to be very slow in the axial direction ($dR/dz \ll 1$). Thus, as indicated in Figure 1, the origin of the coordinate system is located somewhat below the die swell region where the variation of the fiber radius is not slow. This assumption has been the basis of most of the studies on fiber spinning and has been supported by several numerical investigations (Fisher et al., 1980; Keunings et al., 1983). For simplicity, we will further assume that only viscous and viscoelastic forces are important; and gravity, inertia, surface and interfacial tensions are neglected.

Under these assumptions, the dimensionless governing equations and boundary conditions are written as follows:

Core layer

$$u^c_r + \frac{1}{r}u^c + w^c_z = 0 \tag{1a}$$

$$p^c_r = GM\left(u^c_{rr} + \frac{1}{r}u^c_r - \frac{1}{r^2}u^c + \epsilon^2 u^c_{zz}\right) \tag{1b}$$

$$\epsilon^2 p^c_z = GM\left(w^c_{rr} + \frac{1}{r}w^c_r + \epsilon^2 w^c_{zz}\right) \tag{1c}$$

Skin layer

$$u^s_r + \frac{1}{r}u^s + w^s_z = 0 \tag{2a}$$

$$p^s_r = \frac{\partial}{\partial r}\tau^s_{rr} + \frac{1}{r}(\tau^s_{rr} - \tau^s_{\theta\theta}) + \epsilon\frac{\partial}{\partial z}\tau^s_{rz} \tag{2b}$$

$$\epsilon p^s_z = \frac{\partial}{\partial r}\tau^s_{rz} + \frac{1}{r}\tau^s_{rz} + \epsilon\frac{\partial}{\partial z}\tau^s_{zz} \tag{2c}$$

At $r = R(z)$

$$u^s - R_z w^s = 0 \tag{3a}$$

$$\epsilon R_z(\tau^s_{rr} - \tau^s_{zz}) + (1 - \epsilon^2 R_z^2)\tau^s_{rz} = 0 \tag{3b}$$

$$p^s = \frac{1}{(1 + \epsilon^2 R_z^2)}(\tau^s_{rr} - 2\epsilon R_z \tau^s_{rz} + \epsilon^2 R_z^2 \tau^s_{zz}) \tag{3c}$$

At $r = h(z)$

$$u^c = u^s, \quad w^c = w^s \tag{4a}$$

$$u^c - h_z w^c = 0 \tag{4b}$$

$$GM\{2\epsilon^2 h_z(u^c_r - w^c_z) + (1 - \epsilon^2 h_z^2)(\epsilon^2 u^c_z + w^c_r)\} = \epsilon^2 h_z(\tau^s_{rr} - \tau^s_{zz}) + \epsilon(1 - \epsilon^2 h_z^2)\tau^s_{rz} \tag{4c}$$

$$\frac{2GM}{(1 + \epsilon^2 R_z^2)}\{u^c_r - h_z(\epsilon^2 u^c_z + w^c_r) + \epsilon^2 h_z^2 w^c_z\} = (p^c - p^s) + \frac{1}{(1 + \epsilon^2 R_z^2)}(\tau^s_{rr} - 2\epsilon h_z \tau^s_{rz} + \epsilon^2 h_z^2 \tau^s_{zz}) \tag{4d}$$

At $r = 0$

$$u^c, w^c, \text{ and } p^c \text{ are bounded.} \tag{5}$$

where the superscripts c and s denote the variables for core layer and skin layer, respectively. u and w are the radial and axial velocity components, and p the isotropic pressure. $R(z)$ and $h(z)$ indicate the locations of the surface and interface, respectively. Subscripts for τ represent rr-, $\theta\theta$-, rz-, or zz-component of the skin-layer stress tensor, and the subscripts for other variables denote partial differentiation.

The scalings for the nondimensionalization were as follows:

$$(r, z) \rightarrow (R_o, L)$$

$$(u, w) \rightarrow \left(\frac{R_o}{L}w_o^{avg}, w_o^{avg}\right)$$

$$p, \tau \rightarrow \frac{F}{\pi R_o^2}$$

where R_o and L are the radius of the fiber at $z = 0$ and the draw span, respectively. F represents the axial force applied at $z = L$ which controls the final radius of the fiber. w_o^{avg} is the average initial velocity defined as the total flow rate divided by the cross-sectional area at $z = 0$, $\pi(R_o)^2$. Under these scalings, both radial and axial velocity components are preserved in the continuity equation, and the pressure gradient, viscous and viscoelastic forces are balanced in the radial components of the momentum equations. These scalings result in the following dimensionless parameters:

Deborah number	$De = \lambda w_o^{avg}/L$
Shear viscosity ratio	$M = \mu^c/\mu^s$
Dimensionless inverse axial tension	$G = \mu^s Q_T/FL$
Scaling parameter	$\epsilon = R_o/L$
Fraction of skin-layer flow rate	$f = Q_s/Q_T$

where λ is the characteristic time of the upper-convected Maxwell fluid. μ^c and μ^s are the shear viscosities of each layer. Q_s and Q_T are the volumetric flow rate of the skin layer and the total flow rate, respectively.

For the skin layer, the constitutive equation of the upper-convected Maxwell fluid gives the following relations for each component of the stress tensor:

$$\tau_{rr}^s + De\left(u^s \frac{\partial}{\partial r}\tau_{rr}^s + w^s \frac{\partial}{\partial z}\tau_{rr}^s - 2u_r^s\tau_{rr}^s - 2\epsilon u_z^s \tau_{rz}^s\right) = 2Gu_r^s \quad (6a)$$

$$\tau_{\theta\theta}^s + De\left(u^s \frac{\partial}{\partial r}\tau_{\theta\theta}^s + w^s \frac{\partial}{\partial z}\tau_{\theta\theta}^s - 2\frac{u^s}{r}\tau_{\theta\theta}^s\right) = 2G\epsilon\frac{u^s}{r} \quad (6b)$$

$$\epsilon\tau_{zz}^s + De\left(\epsilon u^s \frac{\partial}{\partial r}\tau_{zz}^s + \epsilon w^s \frac{\partial}{\partial z}\tau_{zz}^s - 2w_r^s\tau_{rz}^s - 2\epsilon w_z^s \tau_{zz}^s\right) = 2G\epsilon w_z^s \quad (6c)$$

$$\epsilon\tau_{rz}^s + De\left(\epsilon u^s \frac{\partial}{\partial r}\tau_{rz}^s + \epsilon w^s \frac{\partial}{\partial z}\tau_{rz}^s - \epsilon^2 u_z^s\tau_{zz}^s - w_r^s\tau_{rr}^s - \epsilon(u_r^s + w_z^s)\tau_{rz}^s\right) = G(\epsilon^2 u_z^s + w_r^s) \quad (6d)$$

For a complete description of the system, end conditions at $z = 0$ and $z = 1$ need to be specified. Since the choice of the origin of the draw-down region is rather arbitrary, the proper conditions are not evident. Nevertheless, we will assume that the axial velocity profiles at both ends are known as

$$w^c = w_o^c(r),\ w^s = w_o^s(r) \quad \text{at } z = 0 \quad (7a)$$

$$w^c = w_L^c(r),\ w^s = w_L^s(r) \quad \text{at } z = 1 \quad (7b)$$

It should be noted that w_L is achieved by the applied axial tension F at $z = 1$. For the viscoelastic skin layer, it is also necessary to specify the stress state at $z = 0$ which accounts for the strain history of the fluid to that point. Thus, the stress tensor at $z = 0$ is assumed to be known as

$$\tau^s = \tau_o^s(r) \quad \text{at } z = 0 \quad (7c)$$

As no assumption was made about the uniformity of the axial velocity and the stress field across the cross-section of the fiber, radial dependency of the variables was also retained in the end conditions. These boundary conditions are the typical ones that have been commonly used for single-phase fiber spinning problems.

Equations 1a through 7c fully describe the two-phase flow in the draw-down region. Even with the simplifying assumptions we made, the equations may not be solved analytically. As ϵ is assumed to be very small, however, we will be able to obtain an asymptotic solution through the expansions in the small parameter ϵ.

Asymptotic Analysis

According to the standard procedure of asymptotic expansions, all dependent variables, u, w, p, τ^s, $R(z)$, and $h(z)$ are expanded in integral powers of ϵ as

$$\varphi = \varphi^0 + \epsilon\varphi^1 + \epsilon^2\varphi^{(2)} + \epsilon^3\varphi^{(3)} + \cdots \quad (8)$$

The expansions are substituted into the equations and boundary conditions (Eqs. 1–7). Terms with like powers of ϵ are, then, collected to formulate the problems at each order of ϵ. Thus, the leading order problem $O(\epsilon^0)$ is obtained as follows:

$$u_r^{c0} + \frac{1}{r}u^{c0} + w_z^{c0} = 0 \quad (9a)$$

$$p_r^{c0} = MG\left(u_{rr}^{c0} + \frac{1}{r}u_r^{c0} - \frac{1}{r^2}u^{c0}\right) \quad (9b)$$

$$0 = w_{rr}^{c0} + \frac{1}{r}w_r^{c0} \quad (9c)$$

$$u_r^{s0} + \frac{1}{r}u^{s0} + w_z^{c0} = 0 \quad (10a)$$

$$p_r^{s0} = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}^{s0}) - \frac{1}{r}\tau_{\theta\theta}^{s0} \quad (10b)$$

$$0 = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}^{s0}) \quad (10c)$$

At $r = R^0(z)$,

$$u^{s0} - R_z^0 w^{s0} = 0 \quad (11a)$$

$$\tau_r^{s0} = 0 \quad (11b)$$

$$p^{s0} = \tau_{rr}^{s0} \quad (11c)$$

At $r = h^0(z)$,

$$u^{c0} = u^{s0}, \quad w^{c0} = w^{s0} \quad (12a,b)$$

$$u^{c0} - h_z^0 w^{c0} = 0 \quad (12c)$$

$$w_r^{c0} = 0 \quad (12d)$$

$$p^{c0} - p^{s0} = 2GM(u_r^{c0} - h^0{}_z w_r^{c0}) - \tau_{rr}^{s0} \quad (12e)$$

At $r = 0$,

$$|u^{c0}|, |w^{c0}|, |p^{c0}| < \infty \quad (13)$$

where superscript 0 indicates the order of ϵ. As mentioned previously, the subscripts for τ represent each component of the stress tensor, and the subscripts r and z for other variables denote partial differentiation. In writing Eq. 9c, it has been assumed that $MG = o(\epsilon^{-2})$. Since the inverse axial tension G is typically of order unity or smaller, M is allowed to have any value which is smaller than a very large number $o(\epsilon^{-2})$. Therefore, the restriction imposed on the viscosity ratio M is very mild.

The solution procedure for these equations is straightforward. Thus, the details are omitted, and only the results are given as follows:

$$w^{c0} = w^{s0} = W^0(z) \tag{14a,b}$$

$$u^{c0} = u^{s0} = -\tfrac{1}{2} W_z^0 r \tag{15a,b}$$

$$p^{c0} = \int_{h^0}^{R^0} \frac{1}{r}(\tau_{rr}^{s0} - \tau_{\theta\theta}^{s0})\,dr - GMW_z^0 \tag{16a}$$

$$p^{s0} = \tau_{rr}^{s0} - \int_{R^0}^{r} \frac{1}{r}(\tau_{rr}^{s0} - \tau_{\theta\theta}^{s0})\,dr \tag{16b}$$

$$R^0 = \left[\frac{1}{W^0}\right]^{1/2} \qquad h^0 = \left[\frac{1-f}{W^0}\right]^{1/2} \tag{17a,b}$$

where $W^0(z)$ is an arbitrary function of z, which should be determined from higher-order expansions. These results indicate that the axial velocity for both fluids as well as the pressure field of the Newtonian core layer are radially uniform. However, p^{s0} may not be uniform, as it is determined by the rr- and $\theta\theta$-components of the stress tensor which are not necessarily independent of the radial coordinate nor the same.

Through complicated calculations up to $O(\epsilon^2)$, a differential equation for $W^0(z)$ can be derived as

$$\begin{aligned} 2(W^0)^2 \int_{R^0}^{h^0} r\left[\frac{\partial T}{\partial z} - \frac{\partial E}{\partial z}\right] dr + (1-f)\, W^0 \frac{\partial}{\partial z} E(h^0,z) \\ + W_z^0 \{T(R^0,z) - (1-f)\,T(h^0,z)\} \\ = 3GM(1-f)\,\{(W_z^0)^2 - W^0 W_{zz}^0\} \end{aligned} \tag{18a}$$

where T and E are defined as

$$T(r,z) = \tau_{rr}^{s0} - \tau_{zz}^{s0},\; E(r,z) = \int_{R^0}^{r} \frac{1}{r}(\tau_{rr}^{s0} - \tau_{\theta\theta}^{s0})\,dr \tag{18b,c}$$

and the constitutive equation (Eq. 6) gives the zz-, rr-, and $\theta\theta$-components of the stress tensor as

$$(1 - 2DeW_z^0)\,\tau_{zz}^{s0} + De\left\{-\frac{1}{2} W_z^0 r \frac{\partial}{\partial r}(\tau_{zz}^{s0}) + W^0 \frac{\partial}{\partial z}(\tau_{zz}^{s0})\right\} = 2GW_z^0 \tag{18d}$$

$$(1 + DeW_z^0)\,\tau_{ij}^{s0} + De\left\{-\frac{1}{2} W_z^0 r \frac{\partial}{\partial r}(\tau_{ij}^{s0}) + W^0 \frac{\partial}{\partial z}(\tau_{ij}^{s0})\right\} = -GW_z^0 \tag{18e}$$

where the subscript ij is rr or $\theta\theta$. As pointed out, $\tau_{\theta\theta}$ and τ_{rr} are not necessarily the same, although they satisfy the same differential equation. The difference between them, which accounts for the hoop stress, should depend on the initial stress condition (Eq. 7c), and its influence on axial force balance appears in Eq. 18a.

Axial force $\Phi(z)$ can be expressed as

$$\Phi(z) = \int_{h^0}^{R^0} 2\pi r(\tau_{zz}^{s0} - p^{s0})\,dr + \int_0^{h^0} 2\pi r(\tau_{zz}^{c0} - p^{c0})\,dr \tag{19a}$$

Using Eqs. 16a and 16b, it can be shown that Eq. 18a is equivalent to

$$\frac{d\Phi}{dz} = 0 \tag{19b}$$

indicating that the total axial force is constant along the spinline.

These equations (Eqs. 18a–18e) completely define the leading order axial velocity $W^0(z)$ and may be solved numerically with the end conditions (Eqs. 7a, 7b and 7c). Since the axial velocity is independent of the radial coordinate, the end conditions (Eqs. 7a and 7b) may be modified as

$$W^0 = 1 \qquad \text{at } z = 0 \tag{20a}$$

$$W^0 = Dr \qquad \text{at } z = 1 \tag{20b}$$

where Dr is a dimensionless parameter, "draw ratio," which is defined as the ratio of the average velocity at $z = 1$ to that at $z = 0$. Once $W^0(z)$ is determined by Eqs. 18 and 20, all other variables are determined from Eqs. 15 through 17.

Up to $O(\epsilon)$, the rate of deformation tensor and the voticity tensor are shown to be diagonal and zero, respectively. Thus, the stress tensor may be diagonal (Petrie, 1979). As the axial velocity is independent of the radial coordinate, the diagonal components of the rate of deformation tensor are also independent of r. However, there is no indication that the stress tensor of the skin layer should be radially-independent. It is evident that stress tensor depends on r in the die-swell region near the die exit. It may be possible that the radial dependency can be preserved in the draw-down region, where the smallness of ϵ holds, depending upon the rheological property and the strain history of the material. As no *a priori* assumptions were made to derive Eqs. 18a through 18e other than smallness of ϵ and axisymmetry of the flow, they should be capable of describing such flow situations which are not necessarily one-dimensional. Direct numerical integration of Eqs. 1 through 5 will, of course, give a full description of the flow. However, it will be much more complicated than the numerical integration of Eqs. 18a–18e, since it involves the determination of the free boundary. In that sense, Eqs. 18a through 18e may be useful.

Numerical integration of Eq. 18 can be easily performed for various values of f, G, M and De, once the end conditions are specified. However, we will not discuss the numerical calculations. Instead, we will only consider two limiting cases, where the Deborah number of the skin layer is small ($De \ll 1$), and the applied axial tension is very large compared to the characteristic tension of the skin layer ($G/De \ll 1$). For those cases, our analysis can be compared with the existing studies of single-phase flows (Matovich and Pearson, 1969; Denn et al., 1975; Schultz, 1987).

The Case of Small *De*

For the case of a small Deborah number, we will attempt a perturbation expansion in *De* which will make the asymptotic analysis a two-parameter expansion in ϵ and *De* as

$$\varphi = (\varphi^{00} + De\varphi^{01} + De^2\varphi^{02} + \ldots\ldots) + \epsilon(\varphi^{10} + De\varphi^{11} + De^2\varphi^{12} + \ldots\ldots) + \ldots\ldots$$

where the first and second superscripts for φ indicate the order of ϵ and *De*, respectively. For the current analysis we will focus on the first three terms of the expansion. Thus

$$W^0 = W^{00} + DeW^{01} + De^2W^{02}\ldots\ldots$$
$$\text{and } \tau^0 = \tau^{00} + De\,\tau^{01} + De^2\tau^{02} + \ldots\ldots \quad (21a,b)$$

These expansions are substituted into Eqs. 18 and 20, and solutions for each order of *De* are determined according to the standard procedure of perturbation expansions. After a tedious calculation, we obtain

$$W^0 = e^{\alpha z} + \kappa De^2\alpha^2\{1 + (e^{2\alpha} - 1)z - e^{2\alpha z}\}e^{\alpha z} + O(De^3) \quad (22a)$$

where

$$\alpha = \ln Dr, \qquad \kappa = \frac{f}{f + (1 - f)M} \quad (22b,c)$$

Despite the evidence (Eqs. 18b and 18c) that the expansion should be singular, a regular perturbation expansion was performed for the derivation of Eq. 22. Thus it does not satisfy the initial stress condition (Eq. 7c). However, a singular perturbation expansion will provide only an exponentially decaying correction term restricted to a small inner region (or a boundary layer) near the origin, the order of magnitude of which is $O(De)$ (Denn et al., 1975). As the location of the origin is rather ambiguous at the outset, and our interest is in the region sufficiently far from the die exit, the use of a regular perturbation expansion may be justified. The singular nature of the expansion also implies that the flow quickly becomes one-dimensional, as the radial dependency of the skin-layer stress field, manifested in the initial stress condition, is restricted to the small inner region. Thus, the *a priori* assumption of one-dimensional analyses that all variables are radially independent is justified for fluids with a small *De*.

Using Eq. 22, the relationship between the draw ratio, *Dr*, and the dimensionless inverse axial tension *G* can be shown as

$$\{f + (1 - f)M\}G = \frac{1}{3\ln Dr} + \frac{1}{3}\kappa(1 - Dr^2)De^2 + O(De^3) \quad (23)$$

The factor, $\{f + (1 - f)M\}$, replaces μ_s in $G(=\mu^s Q_T/FL)$ with an average viscosity, $\{f\mu_s + (1 - f)\mu_c\}$. Thus, $\{f + (1 - f)M\}G$ represents the ratio of the average viscous force to the applied axial tension.

It is interesting to note that the only difference between Eqs.

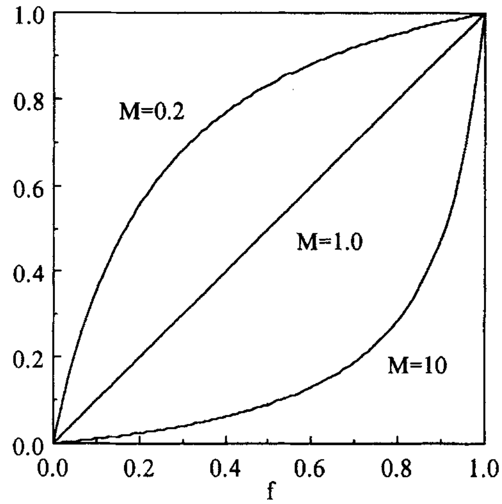

Figure 2. Relative effect of viscosity and flow rate ratios in the limit of small *De*.

22a and 23 and the results for a single-phase flow of an upper-convected Maxwell fluid (Denn et al., 1975) is the factor κ. As its values for various *M* and *f* are given in Figure 2, κ accounts for the relative effects of the viscosity and flow rate ratios. In this limit, the viscosity ratio does not affect the leading-order Newtonian velocity profile, whereas it affects the axial tension required for a given draw ratio.

The $O(De^2)$ term of Eq. 22a is always positive, as described in Figure 3. It provides an especially large correction where the slope of the leading order term is large. Thus, the viscoelasticity of the skin-layer tends to make the radical change of the exponential velocity profile of the Newtonian core layer more gradual. The viscoelasticity effect, however, is suppressed if the viscosity of the Newtonian core layer is large. The second term of Eq. 23 is negative, if $Dr > 1$. Thus, the viscoelasticity of the skin layer has an effect to increase the axial tension that is required to obtain a given draw ratio.

If *M* is zero, we may think that our system is equivalent to a hollow fiber. However, it should be pointed out that it is not the case. The analysis of drawn hollow tubes by Freeman et al. (1986) shows that the internal pressure (p^{c0}), which is an addi-

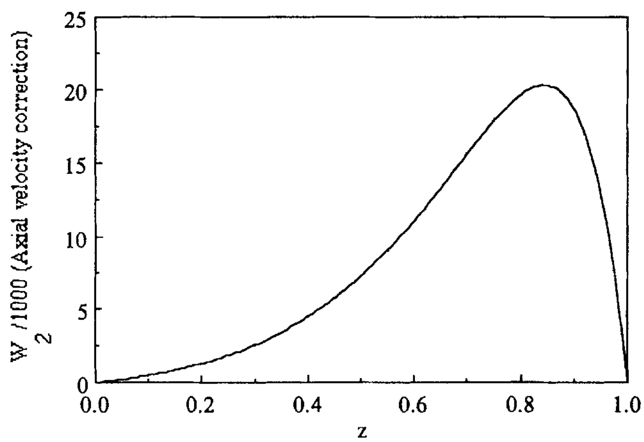

Figure 3. $O(De^2)$ velocity correction ($W_2 = \alpha^2\{1 + (e^{2\alpha} - 1)z - e^{2\alpha z}\}e^{\alpha z}$, $\alpha = 2.996$).

tional control parameter, governs the ratio of inner to outer radius of the hollow tubes. Such important behavior, however, cannot be predicted from our result by simply setting $M = 0$. In our analysis, the core material is incompressible. Consequently, the Eqs. 17a and 17b are forced to hold even for the case of vanishingly small M. Thus, when M is set to zero, our system is reduced only to a special case of zero internal pressure, and the general cases of nonzero internal pressure cannot be predicted.

The Case of a Large Axial Tension

The flow situation with a large applied axial tension is more interesting as it is relevant to industrial practice, and the analysis predicts some important behaviors with practical implications.

From the definitions of G and De, the ratio of the two dimensionless parameters can be shown as

$$\frac{G}{De} = \frac{\mu^s Q_T/FL}{\lambda w_o^{avg}/L} = \frac{\mu^s/\lambda}{F/\pi(R_o)^2}$$

Since μ^s/λ is the elastic shear modulus of the skin layer, G/De represents the ratio of the characteristic stress of the material to the imposed stress to draw a fiber (Denn et al., 1975).

If the applied stress is much larger than the characteristic stress ($G/De \ll 1$), another two-parameter asymptotic expansion is possible (this time in G/De and ϵ). The expansion in this case is not singular, and the initial stress condition (Eq. 7c) may strongly influence the overall flow. The stress condition at $z = 0$, however, is not an experimentally measurable quantity and should be specified rather arbitrarily. In order to avoid time-consuming algebraic complication, we will discuss only a special case of a constant initial stress condition. It should be pointed out, however, that a more general treatment with nonuniform stress conditions also draws similar conclusions.

In the limit of a large applied axial tension ($G/De \approx 0$), Eqs. 18d and 18e become homogeneous. By the method of separation of variables for the homogeneous equations, it is not difficult to show that the stress components should be independent of the radial coordinate to satisfy the constant initial stress condition. Although the extra stress tensor becomes radially uniform in this case, the skin-layer pressure field (Eq. 16d) is a logarithmic function in the radial coordinate since the rr- and $\theta\theta$-components of the extra stress tensor are not generally the same. Thus, the system is not truly a one-dimensional flow even for this special case of a constant initial stress.

Using the fact given in Eq. 19, a simple manipulation of Eqs. 18a through 18e result in the following equations for the leading order in G/De:

$$\tau_{zz}^{s0} = \frac{1}{3f\,De}\frac{W^0}{W_z^0} + \frac{2}{3f}W^0 \qquad (24a)$$

$$\tau_{rr}^{s0} = \frac{2}{3-c}\left(\tau_{zz}^{s0} - \frac{1}{f}W^0\right) \qquad (24b)$$

$$\tau_{\theta\theta}^{s0} = c\,\tau_{rr}^{s0} \qquad (24c)$$

$$2De^2(W_z^0)^3 - De(W_z^0)^2 - W_z^0 + De\,W^0 W_{zz}^0 = 0 \qquad (25)$$

where c is the ratio of $\tau_{\theta\theta}$ to τ_{rr} at $z = 0$. If the zz-component of the initial stress condition is given as t_0, Eq. 24a gives

$$W_z^0 = \frac{1}{De(3t_0 f - 2)} \qquad \text{at } z = 0 \qquad (26)$$

Thus, specifying the initial value of zz-stress component is equivalent to specifying the initial slope of the axial velocity profile. If f is 1 (i.e., single-phase Maxwell fluid spinning), the boundedness of the pressure at $r = 0$ requires c, the ratio $\tau_{\theta\theta}$ to τ_{rr} at $z = 0$, to be one. Thus, the flow becomes fully one-dimensional, and the equations derived by Denn et al. (1975) are recovered. It is interesting to note that Eqs. 25 and 26, which describe the axial velocity profile, are not affected by c. It implies that the hoop stress has no effect on the velocity profile under the given condition, whereas it determines the radial dependency of the skin-layer pressure field.

Using the initial conditions (Eq. 26) and $W^0 = 1$ at $z = 0$, and following a typical transformation for an autonomous differential equation, an analytic solution for Eq. 25 is determined as

$$W^0 + \tfrac{1}{2}k \ln\left[\frac{|W^0 - k|^3\ |1 - k^3|}{|(W^0)^3 - k^3|\ |1 - k|^3}\right] - k\sqrt{3}\left\{\arctan\left(\frac{2W^0 + k}{k\sqrt{3}}\right) - \arctan\left(\frac{2 + k}{k\sqrt{3}}\right)\right\} = 1 + \frac{z}{De} \qquad (27)$$

where

$$k^3 = 1 - (t_0 f)^{-1} \qquad (28)$$

This result is very similar to that of Denn et al. (1975) for a single-phase flow. The only difference is the appearance of f (the flow rate fraction of the skin layer) in k. Despite its striking resemblance, the slight difference manifested in Eq. 28 still provides very interesting features of the two-phase flow.

If the initial slope of the velocity profile is negative (i.e., $t_0 f < 2/3$ from Eq. 26), Eq. 27 shows that W^0 is always smaller than 1. Therefore, $t_0 f$ must be greater than 2/3 in order to achieve $Dr > 1$. In Figure 4 [which is the same as Figure 2 of Denn et al. (1975) with the exception that the parameter t_0 is replaced with $t_0 f$], Eq. 27 is plotted for various values of $t_0 f$ that are greater than 2/3. Like the single-phase Maxwell fluid, the velocity profiles are nearly linear and are relatively insensitive to the value of $t_0 f$ unless it is very large. Figure 4 also indicates that there exists a maximum attainable draw ratio at $z = 1$ for a given Deborah number. Therefore, a skin-layer material, whose Deborah number is smaller than a certain value, should be chosen in order to achieve a given draw ratio. Otherwise, the axial stress becomes infinite before attaining the draw ratio. In Figure 5, the maximum allowable value of De to achieve $Dr = 20$ is plotted against $t_0 f$. $Dr = 20$ was chosen as it represents the critical draw ratio of Newtonian fluids.

The insensitivity of the velocity profile to the value of $t_0 f$ implies that it is also insensitive to f. Thus, in this limit, viscoelasticity effect of the skin layer is dominant and dictates the mechanics of the flow even though its flow rate fraction f may be

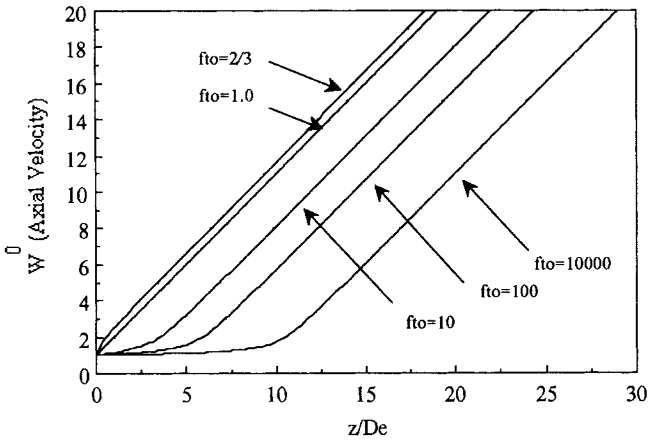

Figure 4. Dimensionless axial velocity in the limit of $G/De \approx 0$.

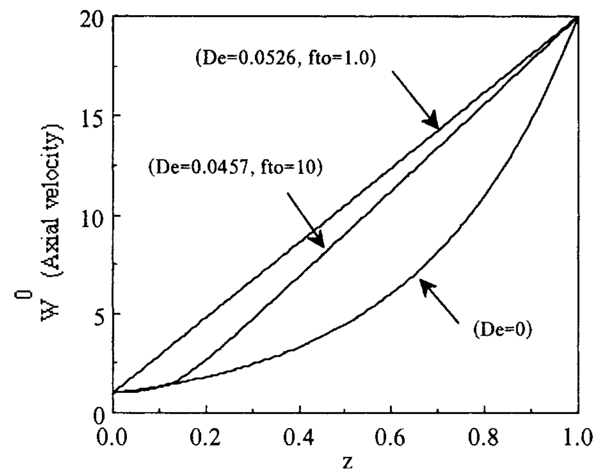

Figure 6. Dimensionless axial velocity profile (Dr = 20).

very small. In Figures 6 and 7, velocity and radius profiles for several different conditions are given. In these figures, $t_0 f$ values of 1 and 10 were chosen as they may represent the typical characteristics of the flow. For each value of $t_0 f$, the maximum allowable De for Dr = 20 can be obtained from Figure 5. It can be noted from the figures that a big difference in the flow rate ratio (0.1 vs. 1.0) does not strongly affect the velocity profile when the applied tension is large. They also imply that for a given skin-layer material (i.e., given De) with a fixed t_0, maximum attainable Dr increases as f decreases. Thus, the maximum attainable draw ratio of a viscoelastic material can be increased if it is coextruded with a less elastic material.

In deriving Eq. 25, it has been assumed that $M(1-f)/f = o[(G/De)^{-1}]$. When considering the restriction imposed on $M[GM = o(\epsilon^{-2})]$, relatively mild restrictions apply to both M and f. Thus, a very small value of f can be allowed even when M is fairly large. This prediction may have a significant implication for industrial applications. If the flow characteristics of a Newtonian-like polymer (e.g., linear polyethylene) are not desirable, it can be significantly modified by coextruding with a small amount of a Maxwell-fluid-like polymer (e.g., branched polyethylene). Since the shear viscosity of the skin layer is allowed to be much smaller than that of the core, it also favors the interfacial stability under a shear flow inside the die. This result is, of course, contingent upon the assumption that the two fluids remain bonded together even when the viscosity difference is large.

We now discuss the contention of Schultz (1987) that there is a serious limitation to the one-dimensional analysis of Denn et al. (1975) for the case of a large applied axial tension ($G/De \ll 1$). His conclusion is based on an order of magnitude analysis in which the results of the leading order approximation $[O(\epsilon^0)]$ were substituted into the z-direction momentum balance (which is equivalent to the Eq. 2c). The results of the one-dimensional analysis are equivalent to the Eqs. 24, 27 and 16b with both f and c equal to one. When t_0 = 1, those equations along with the boundary condition (Eq. 20b) yield

$$W^0 = \tau_{zz}^{s0} = 1 + (Dr - 1)z \quad (29a)$$

$$p^{s0} = \tau_{rr}^{s0} = \tau_{zz}^{s0} - W^0(=0) \quad (29b)$$

Since the viscous-force scaling ($\mu^s W_0^{avg}/L$) was used for the stress tensor in the work by Schultz (1987), Eq. 29a becomes as follows in his scaling:

$$\tau_{zz}^{s0} = \frac{1}{G}\{1 + (Dr - 1)z\} \quad (30)$$

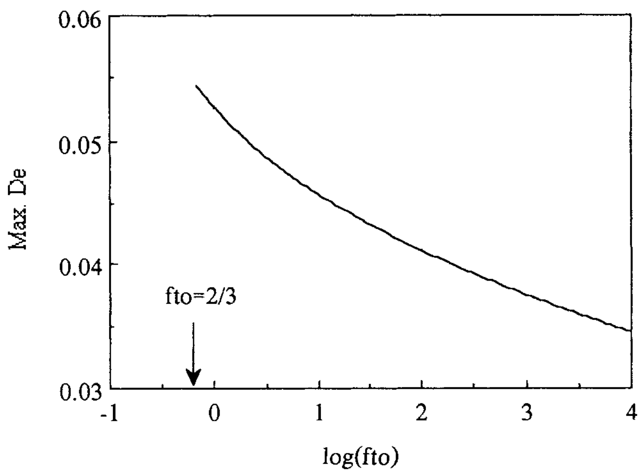

Figure 5. Maximum allowable De as a function of ft_0 (Dr = 20).

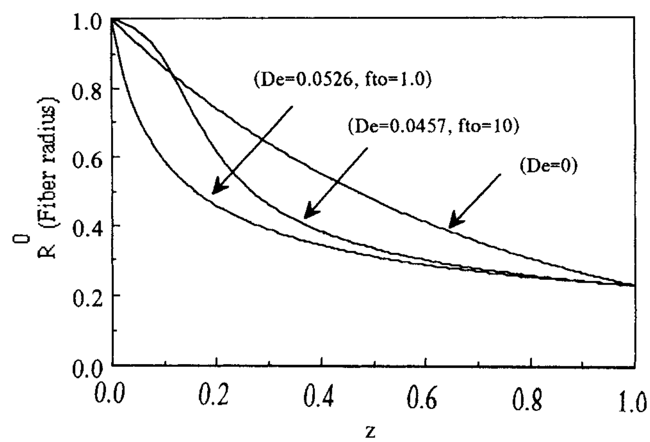

Figure 7. Fiber radius profile (Dr = 20).

He, then, calculates the $O(\epsilon)$ terms in Eq. 2c which become

$$\epsilon\left(p_z^{s0} - \frac{\partial}{\partial z}\tau_{zz}^{s0}\right) = -\frac{\epsilon}{G}(Dr - 1) \tag{31}$$

This term should be small for the asymptotic analysis to be valid. He contends, however, that it is not small for any finite ϵ if applied axial tension is infinitely large (i.e., $G \approx 0$). Thus, he concludes that the one-dimensional theory fails in the limit of a large applied axial tension. It seems, however, that it is an erroneous conclusion caused by an improper scaling of the stress tensor. In Schultz's scalings for pressure, extra stress tensor, and the velocity, Eq. 29b becomes

$$Gp^{s0} = G\tau_{zz}^{s0} - W^0(=0) \tag{32a}$$

If this expression for pressure (although its numerical value is zero) is substituted into Eq. 2c, the $O(\epsilon)$ terms become

$$\epsilon G\left(p_z^{s0} - \frac{\partial}{\partial z}\tau_{zz}^{s0}\right) = \epsilon\left\{\left(G\frac{\partial}{\partial z}\tau_{zz}^{s0} - W_z^0\right) - G\frac{\partial}{\partial z}\tau_{zz}^{s0}\right\} = -\epsilon W_z^0 = -\epsilon(Dr - 1) \tag{32b}$$

which is now negligibly small even in his scalings. If we use $FL/\pi\mu R_0^2$ as the scaling for velocity which is physically inadequate, G, of course, will appear in the denominator of Eq. 32b and the same misleading conclusion may be drawn. The problem analyzed by Schultz (1987) is for the case of an asymptotically small Deborah number in which viscous force is dominant. Thus, the viscous-force scaling is also adequate and should not pose any problem. For the case of a large applied axial tension, however, a physically adequate scaling for the stress tensor is the applied axial tension as was used in our analysis and by Denn et al. (1975). When the correct scaling is used, $1/G$ does not appear in Eq. 31, and the analysis is consistent without any contradiction. Therefore, Schultz's contention that the one-dimensional analysis has serious limitations is apparently incorrect.

Conclusions

Under the assumption that the variation of the fiber radius in the axial direction is very slow, a set of partial differential equations for the mechanics of a two-phase flow has been derived which treats the radial dependency of the stress fields of the viscoelastic skin layer. The analysis is also applicable to the single-phase flows of a Newtonian and an upper-convected Maxwell fluid, as they are two special cases with f equal to zero and one, respectively.

Analytic solutions, which resemble those of the single-phase flows, were obtained for two limiting cases. In the limit of a small Deborah number, the velocity correction to the Newtonian profile is strongly affected by the ratios of shear viscosities and the flow rates of the two fluids. For the case of a large applied axial tension, which may be more relevant to industrial practice, the analysis predicts that

1. Viscoelasticity effect of the skin layer is dominant and dictates the mechanics of the flow even when both of its flow rate and shear viscosity may much smaller than those of the Newtonian core layer.
2. Maximum attainable draw ratio of a viscoelastic material can be increased if it is coextruded with a less elastic material.
3. Hoop stress has no effect on the axial velocity profile if initial stress is radially uniform.

Although our current study does not include stability analysis, we may speculate that the stability of the two-phase flow will be controlled by the viscoelastic skin layer in the limit of a large applied tension. Present analysis is restricted to an axisymmetric flow; nevertheless, the same conclusions for the mechanics of the two-phase flow should be applicable to other two-dimensional flows such as slot cast or extrusion coating flows.

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Notation

De = Deborah number defined as $\lambda W_0^{avg}/L$
Dr = draw ratio
f = fraction of the skin-layer flow rate, Q_s/Q_T
F = axial tension applied at $z = L$
G = dimensionless inverse axial tension
$h(z)$ = location of the interface
L = axial dimension of the draw-down region (draw span)
M = viscosity ratio of the two fluids
p = isotropic pressure
Q_s = flow rate of the skin layer
Q_T = total flow rate
r = radial direction
R_0 = radius of the fiber at $z = 0$
$R(z)$ = location of the surface
t_0 = initial value of τ_{zz}
u = radial velocity component
w = axial velocity component
W_0^{avg} = average axial velocity at $z = 0$
W_L = axial velocity at $z = L$
z = axial direction

Greek letters

α = natural logarithm of Dr
ϵ = scaling parameter defined as R_0/L
θ = azimuthal angle
κ = dimensionless parameter defined as $f/\{f + (1 - f)M\}$
λ = characteristic time of the skin layer (Maxwell fluid)
μ^c, μ^s = shear viscosities of core and skin layers
τ_{ij} = extra stress tensor
τ_0 = extra stress tensor at $z = 0$

Literature Cited

Bell, W. P., and D. D. Edie, "Measured Orientation and Internal Stress Distributions in Melt-Spun Fibers," *J. Appl. Polym. Sci.*, **33**, 1089 (1987).
Chang, H., and A. S. Lodge, "Comparison of Rubberlike-Liquid Theory with Stress-Growth Data for Elongation of a Low-Density Branched Polyethylene Melt," *Rheol. Acta*, **11**, 127 (1972).
Cogswell, F. N., "Tensile Deformation in Molten Polymers," *Rheol. Acta*, **8**, 187 (1969).
Dees, J. R., and J. E. Spruiell, "Structure Development During Melt Spinning of Linear Polyethylene Fibers," *J. Appl. Polym. Sci.*, **18**, 1053 (1974).
Denn, M. M., "Continuous Drawing of Liquids to form Fibers," *Ann. Rev. Fluid Mech.*, **12**, 365 (1980).
Denn, M. M., "Fiber Spinning," *Computational Analysis of Polymer Processing*, J. R. A. Pearson and S. M. Richardson, eds., Applied Science Publishers, London (1983).
Denn, M. M., C. J. S. Petrie, and P. Avenas, "Mechanics of Steady Spinning of a Viscoelastic Liquid," *AIChE J.*, **21**, 791 (1975).
Fisher, R. J., and M. M. Denn, "A Theory of Isothermal Melt Spinning and Draw Resonance," *AIChE J.*, **22**, 236 (1976).
Fisher, R. J., M. M. Denn, and R. I. Tanner, "Initial Profile Development in Melt Spinning," *Ind. Eng. Chem. Fundam.*, **19**, 195 (1980).

Freeman, B. D., M. M. Denn, R. Keunings, G. E. Molau, and J. Ramos, "Profile Development in Drawn Hollow Tubes," *J. Polym. Eng.*, **6**, 171 (1986).

Han, C. D., "A Study of Coextrusion in a Circular Die," *J. Appl. Polym. Sci.*, **19**, 1875 (1975).

Keunings, R., M. J. Crochet, and M. M. Denn, "Profile Development in Continuous Drawing of Viscoelastic Liquids," *Ind. Eng. Chem. Fundam.*, **22**, 347 (1983).

Larson, R. G., "Analysis of Isothermal Fiber Spinning with the Doi-Edwards Constitutive Equation," *J. Rheol.*, **27**, 475 (1983).

Mewis, J., and C. J. S. Petrie, "Hydrodynamics of Spinning Polymers," *Encyclopedia of Fluid Mechanics*, N. P. Cheremisinoff, ed., Vol. 6, Chap. 4, Gulf Publishing, Houston (1987).

Matovich, M. A., and J. R. A. Pearson, "Spinning a Molten Threadline, Steady-State Isothermal Viscous Flows," *Ind. Eng. Chem. Fundam.*, **8**, 512 (1969).

Minoshima, W., and J. L. White, "A Comparative Experimental Study of the Isothermal Shear and Uniaxial Elongational Rheological Properties of Low Density, High Density and Linear Low Density Polyethylenes," *J. Non-Newtonian Fluid Mech.*, **19**, 251 (1986).

Nordberg III, M. E., and H. H. Winter, "Fully Developed Multilayer Polymer Flows in Slits and Annuli," *Polym. Eng. Sci.*, **28**, 444 (1988).

Pearson, J. R. A., and M. A. Matovich, "Spinning a Molten Threadline, Stability," *Ind. Eng. Chem. Fundam.*, **8**, 605 (1969).

Petrie, C. J. S., *Elongatinal Flows*, Pitman, San Francisco (1979).

Petrie, C. J. S., and M. M. Denn, "Instabilities in Polymer Processing," *AIChE J.*, **22**, 209 (1976).

Schultz, W. W., "Slender Viscoelastic Fiber Flow," *J. Rheol.*, **31**, 733 (1987).

Schultz, W. W., and S. H. Davis, "One-Dimensional Liquid Fibers," *J. Rheol.*, **26**, 331 (1982).

Phan-Thien, N., and B. Caswell, "A Nearly-Elongational Flow Analysis of the Fiber Spinning Flow," *J. Non-Newtonian Fluid Mech.*, **21**, 225 (1986).

Spearot, J. A., and A. B. Metzner, "Isothermal Spinning of Molten Polyethylenes," *Trans. Soc. Rheol.*, **16**, 495 (1972).

Waters, N. D., and A. M. Keeley, "The Stability of Two Stratified Non-Newtonian Liquids in Couette Flow," *J. Non-Newtonian Fluid Mech.*, **24**, 161 (1987).

Yeow, Y. L., "On the Stability of Extending Films: a Model for the Film Casting Process," *J. Fluid Mech.*, **66**, 613 (1974).

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